

# Plane wave diffraction in the curved section of the waveguide with a rectangular cross-section

D.Guþas, E.Jotautienė

Lithuanian Agricultural University  
4324 Kaunas

We analyse a normal plane wave travelling along the waveguide with a rectangular cross-section. The pressure in this wave we shall denote by  $p_i$ , the coordinate along the waveguide by  $x_1$ , the time by  $t$ , the circular frequency by  $\omega$  and the wave number by  $k=\omega/c$ . In this case a plane normal wave travelling along the rectilinear section shall be written as:

$$p_i = e^{ikx_1 - i\omega t} \quad (1)$$

This wave, upon reaching the curvilinear section, shall form the reflected field that may be written in the form of the system of plane normal waves

$$p_r = \alpha_0 e^{-ikx_1 - i\omega t} + \sum_1^\infty \alpha_n \cos \frac{n\pi y_1}{a} e^{-ix_1 \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2} - i\omega t} \quad (2)$$

A part of a wave energy penetrates the curvilinear section of the waveguide that we shall describe by means of cylindrical coordinates, and the wave field that passed the curvilinear section we shall connote by  $p_t$ . It is possible to express the pressure in the wave that passed through the curvilinear into the rectilinear part of the semi-waveguide by means of the decomposition into normal waves in the form of

$$p_t = \beta_0 e^{-ikx_2 - i\omega t} + \sum_1^\infty \beta_n \cos \frac{n\pi y_2}{a} e^{+ix_2 \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2} - i\omega t} \quad (3)$$

Here the coordinate  $x_2$  is directed along this rectilinear section as is shown in Fig.1,  $y_2$  is the transversal coordinate,  $a$  is the width.

As it seen, the coefficients  $\alpha_n$  and  $\beta_n$  are the coefficients of reflection and transmission of plane wave through the curvilinear section and the solution of the diffraction problem leads to their determination (with

$$J_n \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2}.$$

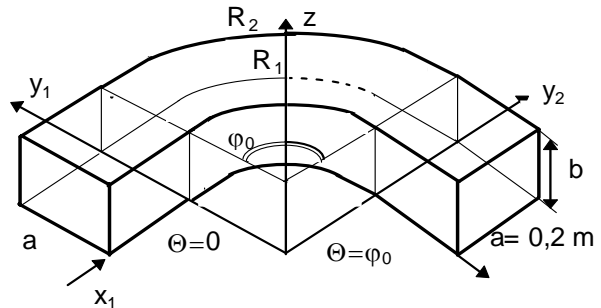


Fig.1. Waveguide with a rectangular cross-section in the curved section

However, for solving of the problem it is more convenient instead of the system of the coefficients  $\alpha_n$  and  $\beta_n$  to examine the distribution of pressure  $p_1(y)$  and velocity  $v_1(y)$  in the plane at the joint of the first semi-waveguide and curvilinear section, as well as pressure  $p_2(y)$  and velocity  $v_2(y)$  at the joint between the curvilinear section and the second semiwaveguide.

The normal velocity of particles in the first cross-section may be expressed by a pressure. Since

$$\rho^0 \omega^2 v_1(y) = \frac{\partial p_1}{\partial x_1} \quad (4)$$

and, on the other side,

$$\frac{\partial p_1}{\partial x} = ik(1 - \alpha_0) + \sum_1^\infty \alpha_n \left[ -i \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2} \right] \cos \frac{n\pi y}{a} \quad (5)$$

then, with the help of Fourier's theorem, we shall obtain

$$1 - \alpha_0 = \frac{\rho \omega^2}{aik} \int_0^a v_1(y) dy \quad (6)$$

$$\alpha_n = \frac{2\rho \omega^2}{a \left[ -i \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2} \right]} \int_0^a v_1(y) \cos \frac{n\pi y}{a} dy,$$

and, consequently

$$\frac{\partial p_1}{\partial x_1} = \rho \omega^2 v_1(y),$$

$$p_1(y) = 2 - \int_0^a v_1(y_1) \times$$

$$\times \left\{ \frac{\rho\omega^2}{ika} + \sum_1^\infty \frac{2\rho\omega^2}{i\sqrt{(ka)^2 - (n\pi)^2}} \cos \frac{n\pi y}{a} \cos \frac{n\pi y_1}{a} \right\} dy_1. \quad (7)$$

Similarly, instead of the system of transmission coefficients  $\beta_n$  we shall introduce the functions  $p_2(y)$  and  $v_2(y)$  (pressure and velocity distribution) in the second cross-section, i.e., at the joint of the curvilinear section and the second semi-waveguide. From Euler's equation

$$\frac{\partial p_2}{\partial x_2} = \rho\omega^2 v_2(y), \quad (8)$$

we shall find all the coefficients

$$\beta_0 = \frac{\rho\omega^2}{ika} \int_0^a v_2(y) dy, \quad (9)$$

$$\beta_n = \frac{2\rho\omega^2}{i\sqrt{(ka)^2 - (n\pi)^2}} \int_0^a v_2(y) \cos \frac{n\pi y}{a} dy.$$

and, finally, we shall expressed

$$\frac{\partial p_2}{\partial x_2} = \rho\omega^2 v_2(y),$$

$$p_2(y) = \int_0^a v_2(y_1) \left[ \frac{\rho\omega^2}{ika} + \sum_1^\infty \frac{2\rho\omega^2 \cos \frac{n\pi y}{a} \cos \frac{n\pi y_1}{a}}{i\sqrt{(ka)^2 - (n\pi)^2}} \right] dy_1. \quad (10)$$

The equations (7)-(10) make it possible to describe the boundary conditions for the equation of the wave field in the curvilinear section of the waveguide. We search for the field in this section as the solution of a wave equation in the cylindrical coordinates  $(r, \theta)$ .

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + k^2 p = 0 \quad (11)$$

at the following boundary conditions

$$\left. \begin{aligned} p(r, 0) &= p_1(r - R_1) \\ \frac{1}{r} \frac{\partial p}{\partial \theta} &= \rho\omega^2 v_1(r - R_1) \end{aligned} \right\}, \quad \theta = 0 \quad (12)$$

$$\left. \begin{aligned} p(r, \varphi_0) &= p_2(r - R_1) \\ \frac{1}{r} \frac{\partial p}{\partial \theta} &= \rho\omega^2 v_2(r - R_1) \end{aligned} \right\}, \quad \theta = \varphi_0$$

Here  $\varphi_0$  is the sprading angle of the curvilinear section,  $R_1$  is the interval radius of bend,  $R_1 + a$  is the external radius. As it seen, the boundary conditions for the field  $p(r, \theta)$  in the essence are the condition for joining wave fields at the different sections of the composite waveguide.

The solution of a wave equation for the curvilinear section may be express by means of the separation of the

variables in the form of the following series

$$p(r, \theta) = \sum_{n=1}^\infty p(\gamma_n, r) \left\{ \frac{a_n \cos \gamma_n (\theta - \varphi_0) - b_n \cos(\gamma_n \theta)}{\gamma_n \sin(\gamma_n \theta_0)} \right\} \quad (13)$$

Here  $\gamma_n$  is the separation constant of the variables which form the series of eigenvalues [I]. From the wave equation (12) we obtain the following equation

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} p(\gamma_n, r) + \left( k^2 - \frac{\gamma_n^2}{r^2} \right) p(\gamma_n, r) = 0 \quad (14)$$

The exact solution of this equation may be given in terms of the Bessel's  $J_\gamma$  ( $kr$ ) and Neumann's  $N_\gamma$  ( $kr$ ) functions

$$p(\gamma_n, r) = J_{\gamma_n}(kr) \dot{N}_{\gamma_n}(kR_1) - N_{\gamma_n}(kr) \dot{J}_{\gamma_n}(kR_1) \quad (15)$$

where  $\gamma_n$  is the solution of the frequency equation

$$\dot{J}_{\gamma_n}(kR_2) \dot{N}_{\gamma_n}(kR_1) - \dot{N}_{\gamma_n}(kR_2) \dot{J}_{\gamma_n}(kR_1) = 0 \quad (16)$$

which also comprises complex numbers

$$\gamma_n = i\gamma'_n$$

These forms possess the properties of orthogonality

$$\int_{R_1}^{R_2} p(\gamma_n, r) p(\gamma_m, r) r^{-1} dr = 0, \quad \int_{R_1}^{R_2} p^2(\gamma_n, r) r^{-1} dr = N_n.$$

where  $N_n$  is the essence of the norms of eigenforms.

The selected form of solution (13) fully corresponds to the physics of the phenomenon, i.e., permits the transition to the travelling and reflected waves of the type  $a_n e^{i\gamma_n \theta} + b_n e^{-i\gamma_n \theta}$

Now we shall start fulfilling the boundary conditions at the boundaries of the division between the rectilinear sections and the curved part of the waveguide. From the boundary conditions (12) with the help of solution (13) we shall get four equations

$$\frac{1}{r} \sum_{n=1}^\infty a_n p(\gamma_n, r) = \rho\omega^2 v_1(r - R_1),$$

$$\frac{1}{r} \sum_{n=1}^\infty b_n p(\gamma_n, r) = \rho\omega^2 v_2(r - R_1),$$

$$\sum_{n=1}^\infty \left\{ \frac{a_n \cos(\gamma_n \varphi_0) - b_n}{\gamma_n \sin(\gamma_n \varphi_0)} \right\} p(\gamma_n, r) = p_1(r - R_1), \quad (17)$$

$$\sum_{n=1}^\infty \left\{ \frac{a_n - b_n \cos(\gamma_n \varphi_0)}{\gamma_n \sin(\gamma_n \varphi_0)} \right\} p(\gamma_n, r) = p_2(r - R_1).$$

At first it is more convenient to analyse the two last equations. By means of the integral transform with the account of the orthogonality of forms  $p(\gamma_n, r)$  we shall obtain

$$a_n \cos(\gamma_n \varphi_o) - b_n = \frac{\gamma_n \sin(\gamma_n \varphi_o)}{N_n} \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr,$$

$$a_n - \cos(\gamma_n \varphi_o) b_n = \frac{\gamma_n \sin(\gamma_n \varphi_o)}{N_n} \int_{R_1}^{R_2} p_2(r - R_1) p(\gamma_n, r) r^{-1} dr$$

(18)  
Solving this system of algebraic equations we shall get

$$a_n = \frac{\gamma_n}{N_n \sin(\gamma_n \varphi_o)} \left\{ \int_{R_1}^{R_2} p_2(r - R_1) p(\gamma_n, r) r^{-1} dr - \right.$$

$$\left. - \cos(\gamma_n \varphi_o) \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr \right\}.$$

$$b_n = \frac{\gamma_n}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma_n \varphi_o) \int_{R_1}^{R_2} p_2(r - R_1) \times \right.$$

$$\left. \times p(\gamma_n, r) r^{-1} dr - \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr \right\}$$

(19)  
Substituting these solutions in the first pair of equations from formulas (13) we shall get

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ p_2(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) - \right.$$

$$\left. - \cos(\gamma_n \varphi_o) p_1(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) \right\} =$$

$$= \rho \omega^2 v_1(r - R_1),$$

(20)

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma_n \varphi_o) p_2(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) - \right.$$

$$\left. - p_1(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho \omega^2 v_2(r - R_1).$$

The values  $p_1(r - R_1)$  and  $p_2(r - R_1)$  are self-expressed by  $v_1(y)$  and  $v_2(y)$

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ p(r_n, r_1) p(\gamma_n, r) \int_0^a v_2(y_1) \times \right.$$

$$\times \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 +$$

$$+ \cos(\gamma_n \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \int_0^a v_1(y_1) \left[ \frac{\rho \omega^2}{ika} + \right.$$

$$+ \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \left. \right] dy_1 -$$

$$\left. - 2 \cos(\gamma_n \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho \omega^2 v_1(r - R_1),$$

(21)

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma_n, \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \times \right.$$

$$\times \int_0^a v_2(y_1) \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 +$$

$$+ p(\gamma_n, r_1) p(\gamma_n, r) \int_0^a v_1(y_1) \left[ \frac{\rho \omega^2}{ika} + \right.$$

$$+ \sum_{m=1}^{\infty} \frac{\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \left. \right] dy_1 -$$

$$\left. - 2 p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho \omega^2 - v_2(r - R_1).$$

Commonly similar systems of Fredholm's integral equations we shall describe in the more compact form with the use of the concept of the kernel of integral transform. This system actually is given in the following sufficiently compact form

$$\int_0^a v_1(y_1) K_{11}(y_1, r) dy_1 + \int_0^a v_2(y_1) \times$$

(22)

$$\times K_{12}(y_1, r) dy_1 - \rho \omega^2 v_1(r - R_1) = \Phi_1(r)$$

$$\int_0^a v_1(y_1) K_{21}(y_1, r) dy_1 + \int_0^a v_2(y_1) \times$$

$$\times K_{22}(y_1, r) dy_1 - \rho \omega^2 v_2(r - R_1) = \Phi_2(r)$$

where the following designations are introduced.

$$K_{11}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right],$$

$$K_{12}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1 \cos(\gamma_n \varphi_o)}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right],$$

(23)

$$K_{21}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right],$$

$$K_{22}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1 \cos(\gamma_n \varphi_0)}{N_n \sin(\gamma_n \varphi_0)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[ \frac{\rho \omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho \omega^2 \cos \frac{m\pi(r_1 - R_1)}{a} \cos \frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right].$$

$$\Phi_1 = 2 \int_{R_1}^{R_2} \sum_{n=1}^{\infty} p(\gamma_n, r_1) p(\gamma_n, r) \frac{\gamma_n r^{-1} r_1^{-1} \cos(\gamma_n \varphi_0) dr_1}{N_n \sin(\gamma_n \varphi_0)},$$

$$\Phi_2 = 2 \int_{R_1}^{R_2} \sum_{n=1}^{\infty} p(\gamma_n, r_1) p(\gamma_n, r) \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_0)}.$$

We had obtained the integral equations that describe the plane wave diffraction.

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D.Guþas, E.Jotautienė

**Plokðëiosios bangos difrakcija iðlenktoje staèiakampio skerspjûvio bangolaidþio dalyje**

Reziiumė

Darbo tikslas - gauti integraliniø lygëiø sistemà plokðëiosios bangos difrakcijai apibrëptomis sąlygomis apskaièiuoti.

Naudotasi Ðturmo-Liuvilio metodu. Sprendimo ieðkota tiriant normalià plokðëiàjà bangà, sklindanèià staèiakampio formos bangolaidþiu.

Banginës energijos sklidimas bangolaidþio iðlenktàja dalmi, tirtas cilindrinëse koordinatëse, kai kraðtinës sąlygos formuotos tarp tiesiosios ir iðlenktosios bangolaidþio dalies.

Sprendþiant banginæ lygtà sudaryta lygëiø sistema. Jà iðsprendus gauta plokðëiosios bangos difrakcija alkûnëje.

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