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Plane wave diffraction in the curved section of the waveguide with a rectangular cross-section

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We analyse a normal plane wave travelling along the waveguide with a rectangular cross-section. The pressure in this wave we shall denote by p_i, the coordinate along the waveguide by x_1, the time by t, the circular frequency by ω and the wave number by $k=\omega/c$.
In this case a plane normal wave travelling along the rectilinear section shall be written as:

$$p_i = e^{ikx_1 - i\omega t} . \qquad (1)$$

This wave, upon reaching the curvilinear section, shall form the reflected field that may be written in the form of the system of plane normal waves

$$p_r = \alpha_0 e^{-ikx_1 - i\omega t} + \sum_1^{\infty} \alpha_n \cos\frac{n\pi y_1}{a} e^{-ix_1\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2} - i\omega t} . \qquad (2)$$

A part of a wave energy penetrates the curvilinear section of the waveguide that we shall describe by means of cylindrical coordinates, and the wave field that passed the curvilinear section we shall connote by p_t. It is possible to express the pressure in the wave that passed through the curvilinear into the rectilinear part of the semi-waveguide by means of the decomposition into normal waves in the form of

$$p_t = \beta_0 e^{-ikx_2 - i\omega t} + \sum_1^{\infty} \beta_n \cos\frac{n\pi y_2}{a} e^{+ix_2\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2} - i\omega t} . \qquad (3)$$

Here the coordinate x_2 is directed along this rectilinear section as is shown in Fig.1, y_2 is the transversal coordinate, a is the width.

As it seen, the coefficients α_n and β_n are the coefficients of reflection and transmission of plane wave through the curvilinear section and the solution of the diffraction problem leads to their determination (with $J_m\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2}$).

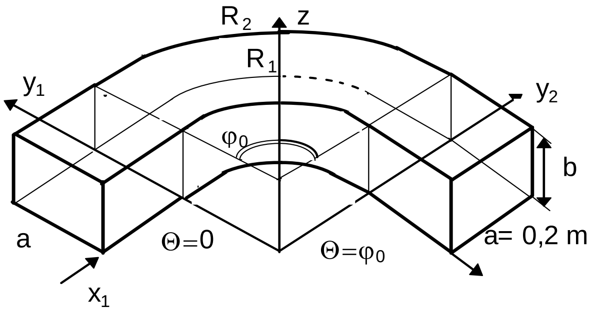

Fig.1.**Waveguide with a rectangular cross-section in the curved section**

However, for solving of the problem it is more convenient instead of the system of the coefficients α_n and β_n to examine the distribution of pressure $p_1(y)$ and velocity $v_1(y)$ in the plane at the joint of the first semi-waveguide and curvilinear section, as well as pressure $p_2(y)$ and velocity $v_2(y)$ at the joint between the curvilinear section and the second semiwaveguide.

The normal velocity of particles in the first cross-section may be expressed by a pressure. Since

$$\rho^0\omega^2 v_1(y) = \frac{\partial p_1}{\partial x_1} , \qquad (4)$$

and, on the other side,

$$\frac{\partial p_1}{\partial x} = ik(1-\alpha_0) + \sum_1^{\infty} \alpha_n \left[-i\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2} \right] \cos\frac{n\pi y}{a} , \qquad (5)$$

then, with the help of Fourier's theorem, we shall obtain

$$1-\alpha_0 = \frac{\rho\omega^2}{aik} \cdot \int_0^a v_1(y)dy, \qquad (6)$$

$$\alpha_n = \frac{2\rho\omega^2}{a\left[-i\sqrt{k^2-\left(\frac{n\pi}{a}\right)^2}\right]} \int_0^a v_1(y)\cos\frac{n\pi y}{a} dy ,$$

and, consequently

$$\frac{\partial p_1}{\partial x_1} = \rho\omega^2 v_1(y),$$

$$p_1(y) = 2 - \int_0^a v_1(y_1) \times$$

$$\times\left\{\frac{\rho\omega^2}{ika}+\sum_1^{\infty}\frac{2\rho\omega^2}{i\sqrt{(ka)^2-(n\pi)^2}}\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}\right\}dy_1 . \quad (7)$$

Similarly, instead of the system of transmission coefficients β_n we shall introduce the functions $p_2(y)$ and $v_2(y)$ (pressure and velocity distribution) in the second cross-section, i.e., at the joint of the curvilinear section and the second semi-wavequide. From Euler's equation

$$\frac{\partial p_2}{\partial x_2}=\rho\omega^2 v_2(y) , \quad (8)$$

we shall find all the coefficients

$$\beta_o=\frac{\rho\omega^2}{ika}\int_o^a v_2(y)dy,$$

$$\beta_n=\frac{2\rho\omega^2}{i\sqrt{(ka)^2-(n\pi)^2}}\int_o^a v_2(y)\cos\frac{n\pi y}{a}dy. \quad (9)$$

and, finally, we shall expressed

$$\frac{\partial p_2}{\partial x_2}=\rho\omega^2 v_2(y),$$

$$p_2(y)=\int_o^a v_2(y_1)\left[\frac{\rho\omega^2}{ika}+\sum_1^{\infty}\frac{2\rho\omega^2\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}}{i\sqrt{(ka)^2-(n\pi)^2}}\right]dy_1. \quad (10)$$

The equations (7)-(10) make it possible to decribe the boundary conditions for the equation of the wave field in the curvilinear section of the waveguide. We search for the field in this section as the solution of a wave equation in the cylindrical coordinates (r, θ).

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}+k^2 p=0 \quad (11)$$

at the following boundary conditions

$$\left.\begin{aligned}&p(r,o)=p_1(r-R_1)\\&\frac{1}{r}\frac{\partial p}{\partial\theta}=\rho\omega^2 v_1(r-R_1)\end{aligned}\right\} , \ \theta=0$$

$$\left.\begin{aligned}&p(r,\varphi_o)=p_2(r-R_1)\\&\frac{1}{r}\frac{\partial p}{\partial\theta}=\rho\omega^2 v_2(r-R_1)\end{aligned}\right\} , \ \theta=\varphi_o \quad (12)$$

Here φ_o is the sprading angle of the curvilinear section, R_1 is the interval radius of bend, R_1+a is the external radius. As it seen, the boundary conditions for the field $p(r,\theta)$ in the essence are the condition for joining wave fields at the different sections of the composite waveguide.

The solution of a wave equation for the curvilinear section may be express by means of the separation of the variables in the form of the following series

$$p(r,\theta)=\sum_{n=1}^{\infty}p(\gamma_n,r)\left\{\frac{a_n\cos\gamma_n(\theta-\varphi_o)-b_n\cos(\gamma_n\theta)}{\gamma_n\sin(\gamma_n\theta_o)}\right\} \quad (13)$$

Here γ_n is the separation constant of the variables which form the series of eigenvalues [I]. From the wave equation (12) we obtain the following equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}p(\gamma_n,r)+\left(k^2-\frac{\gamma_n^2}{r^2}\right)p(\gamma_n,r)=0 \quad (14)$$

The exact solution of this equation may be given in terms of the Bessel's $J_\gamma(kr)$ and Neumann's $N_\gamma(kr)$ functions

$$p(\gamma_n,r)=J_{\gamma_n}(kr)\dot{N}_{\gamma_n}(kR_1)-N_{\gamma_n}(kr)\dot{J}_{\gamma_n}(kR_1) \quad (15)$$

where γ_n is the solution of the frequency equation

$$\dot{J}_{\gamma_n}(kR_2)\dot{N}_{\gamma_n}(kR_1)-\dot{N}_{\gamma_n}(kR_2)\dot{J}_{\gamma_n}(kR_1)=0 \quad (16)$$

which also comprises complex numbers $\gamma_n=i\gamma_n'$

These forms possess the properties of orthogonality

$$\int_{R_1}^{R_2}p(\gamma_n,r)p(\gamma_m,r)r^{-1}dr=0,\ \int_{R_1}^{R_2}p^2(\gamma_n,r)r^{-1}dr=N_n.$$

where N_n is the essence of the norms of eigenforms.

The selected form of solution (13) fully corresponds to the physics of the phenomenon, i.e., permits the transition to the travelling and reflected waves of the type $a_n'e^{i\gamma_n\theta}+b_n'e^{-i\gamma_n\theta}$

Now we shall start fulfilling the boundary conditions at the boundaries of the division between the rectilinear sections and the curved part of the waveguide. From the boundary conditions (12) with the help of solution (13) we shall get four equations

$$\frac{1}{r}\sum_{n=1}^{\infty}a_n p(\gamma_n,r)=\rho\omega^2 v_1(r-R_1),$$

$$\frac{1}{r}\sum_{n=1}^{\infty}b_n p(\gamma_n,r)=\rho\omega^2 v_2(r-R_1),$$

$$\sum_{n=1}^{\infty}\left\{\frac{a_n\cos(\gamma_n\varphi_o)-b_n}{\gamma_n\sin(\gamma_n\varphi_o)}\right\}p(\gamma_n,r)=p_1(r-R_1), \quad (17)$$

$$\sum_{n=1}^{\infty}\left\{\frac{a_n-b_n\cos(\gamma\varphi_o)}{\gamma_n\sin(\gamma_n\varphi_o)}\right\}p(\gamma_n,r)=p_2(r-R_1).$$

At first it is more convenient to analyse the two last equations. By means of the integral transformat with the account of the orthogonality of forms $p(\gamma_n, r)$ we shall obtain

$$a_n \cos(\gamma_n \varphi_o) - b_n = \frac{\gamma_n \sin(\gamma \varphi_o)}{N_n} \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr,$$

$$a_n - \cos(\gamma_n \varphi_o) b_n = \frac{\gamma_n \sin(\gamma \varphi_o)}{N_n} \int_{R_1}^{R_2} p_2(r - R_1) p(\gamma_n, r) r^{-1} dr \qquad (18)$$

Solving this system of algebraic equations we shall get

$$a_n = \frac{\gamma_n}{N_n \sin(\gamma_n \varphi_o)} \left\{ \int_{R_1}^{R_2} p_2(r - R_1) p(\gamma_n, r) r^{-1} dr - \right.$$

$$\left. - \cos(\gamma_n \varphi_o) \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr \right\}.$$

$$b_n = \frac{\gamma_n}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma, \varphi_o) \int_{R_1}^{R_2} p_2(r - R_1) \times \right.$$

$$\left. \times p(\gamma_n, r) r^{-1} dr - \int_{R_1}^{R_2} p_1(r - R_1) p(\gamma_n, r) r^{-1} dr \right\} \qquad (19)$$

Substituting these solutions in the first pair of equations from formulas (13) we shall get

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^1 r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ p_2(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n\ , r) - \right.$$

$$\left. - \cos(\gamma_n \varphi_o) p_1(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) \right\} =$$

$$= \rho\omega^2 v_1(r - R_1), \qquad (20)$$

$$\int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma_n \varphi_o) p_2(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) - \right.$$

$$\left. - p_1(r_1 - R_1) p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho\omega^2 v_2(r - R_1),$$

The values $p_1(r-R_1)$ and $p_2(r-R_1)$ are self-expressed by $v_1(y)$ and $v_2(y)$

$$\int_{R_1}^{R_2} \sum_{I}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1}, dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ p(r_n, r_1) p(\gamma_n, r) \int_o^a v_2(y_1) \times \right.$$

$$\times \left[\frac{\rho\omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 +$$

$$+ \cos(\gamma_n \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \int_o^a v_1(y_1) \left[\frac{\rho\omega^2}{ika} + \right.$$

$$\left. + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 -$$

$$\left. - 2\cos(\gamma_n \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho\omega^2 v_1(r - R_1), \qquad (21)$$

$$\int_{R_1}^{R_2} \sum_{I}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} \left\{ \cos(\gamma_n, \varphi_o) p(\gamma_n, r_1) p(\gamma_n, r) \times \right.$$

$$\times \int_o^a v_2(y_1) \left[\frac{\rho\omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 +$$

$$+ p(\gamma_n, r_1) p(\gamma_n, r) \int_o^a v_1(y_1) \left[\frac{\rho\omega^2}{ika} + \right.$$

$$\left. + \sum_{m=1}^{\infty} \frac{\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y_1}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right] dy_1 -$$

$$\left. - 2 p(\gamma_n, r_1) p(\gamma_n, r) \right\} = \rho\omega^2 - v_2(r - R_1).$$

Commonly similar systems of Fredholm's integral equations we shall decribe in the more compact form with the use of the concept of the kernel of integral transformat. This system actually is given in the following sufficiently compact form

$$\int_o^a v_1(y_1) K_{11}(y_1, r) dy_1 + \int_o^a v_2(y_1) \times$$

$$\times K_{12}(y_1, r) dy_1 - \rho\omega^2 v_1(r - R_1) = \Phi_1(r) \qquad (22)$$

$$\int_o^a v_1(y_1) K_{21}(y_1, r) dy_1 + \int_o^a v_2(y_1) \times$$

$$\times K_{22}(y_1, r) dy_1 - \rho\omega^2 v_2(r - R_1) = \Phi_2(r)$$

where the following designations are introduced.

$$K_{11}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[\frac{\rho\omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right],$$

$$K_{12}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1 \cos(\gamma_n \varphi_o)}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[\frac{\rho\omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right], \qquad (23)$$

$$K_{21}(y, r) = \int_{R_1}^{R_2} \sum_{n=1}^{\infty} \frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n \varphi_o)} p(\gamma_n, r_1) p(\gamma_n, r) \times$$

$$\times \left[\frac{\rho\omega^2}{ika} + \sum_{m=1}^{\infty} \frac{2\rho\omega^2 \cos\frac{m\pi(r_1 - R_1)}{a} \cos\frac{m\pi y}{a}}{i\sqrt{(ka)^2 - (m\pi)^2}} \right],$$

$$K_{22}(y,r)=\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1 \cos(\gamma_n\varphi_o)}{N_n \sin(\gamma_n\varphi_o)}\, p(\gamma_n,r_1)\,p(\gamma_n,r)\times$$

$$\times\left[\frac{\rho\omega^2}{ika}+\sum_{m=1}^{\infty}\frac{2\rho\omega^2\cos\frac{m\pi(r_1-R_1)}{a}\cos\frac{m\pi y}{a}}{i\sqrt{(ka)^2-(m\pi)^2}}\right].$$

$$\Phi_1=2\int_{R_1}^{R_2}\sum_{n=1}^{\infty} p(\gamma_n,r_1)\,p(\gamma_n,r)\frac{\gamma_n r^{-1} r_1^{-1}\cos(\gamma_n\varphi_o)dr_1}{N_n\sin(\gamma_n\varphi_o)},$$

$$\Phi_2=2\int_{R_1}^{R_2}\sum_{n=1}^{\infty} p(\gamma_n,r_1)\,p(\gamma_n,r)\frac{\gamma_n r^{-1} r_1^{-1}dr_1}{N_n\sin(\gamma_n\varphi_o)}.$$

We had obtained the integral equations that describe the plane wave diffraction.

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Plokðèiosios bangos difrakcija iðlenktojoje staèiakampio skerspjûvio bangolaidþio dalyje

Reziumë

Darbo tikslas - gauti integraliniø lygèiø sistemà plokðèiosios bangos difrakcijai apibrëþtomis sàlygomis apskaièiuoti.

Naudotasi Ðturmo-Liuvilio metodu. Sprendimo ieðkota tiriant normalià plokðèiàjà bangà, sklindanèià staèiakampio formos bangolaidþiu.

Banginës energijos sklidimas bangolaidþio iðlenktàja dalmi, tirtas cilindrinëse koordinatëse, kai kraðtinës sàlygos formuotos tarp tiesiosios ir iðlenktosios bangolaidþio dalies.

Sprendþiant banginæ lygtá sudaryta lygèiø sistema. Jà iðsprendus gauta plokðèiosios bangos difrakcija alkûnëje.