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Normal wave diffraction in the rounded section of a waveguide

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Introduction

An issue of sound wave propagation in curved ducts has attracted attention of numerous researchers. The problem is of serious interest since it is the generalization of the theory of waveguides, Solution of problems related to wave propagation in curved ducts is of special practical importance, since almost any piping system includes conjugations of straight sections by means of curved ones. Normal wave propagation in the curved bend section of the waveguide will be studied using methods analogous to those as in our paper [1]. Plane wave propagation in the elbow bend was investigated in [1]. Plane wave sound energy is directed towards longitudinal axis along the front.

Therefore it is reasonable to analyse issues concerning curved ducts with regard to the application of mathematical methods for normal waves.

It should be noted that there are other methods for computation of sound propagation in curvilinear waveguides. For example, in [2,3] a general method for studying of heterogeneous waveguides - a method of transversal cross section - is applied. However, in the general case it is a rather bulky algorithm, requiring the solution of an infinite system of differential equations.

Theory of normal wave diffraction in curved ducts

Further we shall study a normal wave propagating along a waveguide with a constant cross-section. Due to the separability of variables in both rectangular and cylindrical coordinates it is possible to see the single wave of m-th order as an initial exciting field:

$$p_i = \cos\frac{m\pi z}{b} e^{ix_1\sqrt{K^2-\left(\frac{m\pi}{b}\right)^2}-i\omega t} \quad (1)$$

Further the *m* index shall be omitted, taking into account the separability of the corresponding coordinates and accepting the above-introduced notations. Thus the diffracted field in the semi-finite rectilinear section p_2 shall be written without the index *m* in the form of

$$p_2 = \alpha_0 e^{-iK_1x_1-i\omega t} + \sum_1^\infty \alpha_n \cos\frac{n\pi y}{a} e^{-ix_1\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}-i\omega t} \quad (2)$$

and the one passing through the rounded section of the part in the form of p_t

$$p_t = \beta_0 e^{iK_2x_2-i\omega t} + \sum_1^\infty \beta_n \cos\frac{n\pi y_2}{a} e^{ix_2\sqrt{K_2^2-\left(\frac{n\pi}{a}\right)^2}-i\omega t}, \quad (3)$$

$$K_2 = K_1 = \sqrt{\frac{\omega^2}{c^2}-\left(\frac{m\pi}{b}\right)^2}.$$

Here the above-introduced notations and the condition

$$\frac{\omega^2}{c^2} > \frac{m^2\pi^2}{b^2}, \qquad J_m\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2} > 0. \quad (4)$$

are used, which expresses the possibility of realization of the *m*-th form in the semi-infinite waveguide, i.e., the non-zero condition of excitation.

As in the previous paper instead of the coefficients we shall search for $p_1(y)$ and $v_1(y)$. From the boundary conditions at the joint of the curvilinear and the first rectilinear sections

$$\rho_1\omega^2 v_1(y) = \frac{\partial p_1}{\partial x_1} \quad (5)$$

and on the other side

$$\frac{\partial p_1}{\partial x} = iK_1(1-\alpha_0) + \sum_1^\infty \alpha_n\left[-i\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}\right]\cos\frac{n\pi y}{a} \quad (6)$$

then with the help of Fourier's theorem we shall obtain

$$1-\alpha_0 = \frac{\rho_1\omega^2}{iK_1a}\int_0^a v_1(y)dy,$$

$$\alpha_n = \frac{2\rho\omega^2}{a\left[-i\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}\right]}\int_0^a v_1(y)\cos\frac{n\pi y}{a}dy,$$

and, consequently

$$\frac{\partial p_1}{\partial x_1} = \rho\omega^2 v_1(y,) \quad (7)$$

$$p_1(y)=2-\int_0^a v_1(y)\left\{\frac{\rho\omega^2}{iK_1a}+\sum_1^\infty\frac{2\rho\omega^2}{i\sqrt{K_1^2a^2-(n\pi)^2}}\times\right.$$
$$\left.\times\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}\right\}dy.$$

In an analogous way, instead of the system of transmission coefficients β_n we shall introduce the functions $p_2(y)$ and $v_2(y)$, which stand for the distribution of the pressure and the velocity of particles in the second cross-section, i.e., at the joint of the curvilinear section and the second semi-waveguide. From Euler's equation [1] we shall find coefficients β_n, expressing them in terms of $v_2(y)$:

$$\beta_0=\frac{\rho\omega^2}{iK_1a}\int_0^a v_2(y)dy, \quad (8)$$

$$\beta_n=\frac{2\rho\omega^2}{i\sqrt{(K_1a)^2-(n\pi)^2}}\int_0^a v_2(y)\cos\frac{n\pi y}{a}dy,$$

and

$$\frac{\partial p_2}{\partial x_2}=\rho\omega^2 v_2(y), \quad (9)$$

$$p_2(y)=\int_o^a v_2(y)\left[\frac{\rho\omega^2}{iK_1a}+\sum_1^\infty\frac{2\rho\omega^2\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}}{i\sqrt{(K_1a)^2-(n\pi)^2}}\right]dy_1.$$

The equations (7)-(9) enable the boundary conditions for a problem on a wave field in the curvilinear section of a waveguide to be defined. The field in that section must satisfy a wave equation, given in the cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}+K_1^2p=0 \quad (10)$$

at the following boundary conditions

$$\left.\begin{aligned}p(r,0)&=p_1(r-R_1)\\ \frac{1}{r}\frac{\partial p}{\partial\theta}&=\rho\omega^2v(r-R_1)\end{aligned}\right\}\theta=0,$$
$$\left.\begin{aligned}p(r,\varphi_0)&=p_2(r-R_1)\\ \frac{1}{r}\frac{\partial p}{\partial\theta}&=\rho\omega^2v_2(r-R_1)\end{aligned}\right\}\theta=\varphi_0. \quad (11)$$

Here φ_0 is the angular width the curvilinear section, R_1 is the inferior of the rounding radii, $R_2=R_1+a$ is the outer radius of the rounded part of the waveguide.

As is seen, the boundary conditions for the field $p(r,\theta,z)$, in the essence, are the conditions for the joining of the wave fields on the boundaries of the rectilinear and rounded sections of the waveguide.

The solution of the wave equation (10) for the curvilinear section may be described by means of the separation of the variables in the form of the following series

$$p(r,\theta)=\sum_{n=0}p(\gamma_nr)\left\{\frac{a_n\cos\gamma_n(\theta-\varphi_0)-b_n\cos\gamma_n\theta}{\gamma_n\sin\gamma_n\varphi_0}\right\}$$
(12)

The multiplier $\cos\frac{mz\pi}{b}$ as it was mentioned, shall be omitted everywhere. Here γ_n are the constant separations of the variables that are forming the series of the eigenvalues [4]. From the wave equation (10) we shall obtain by Fourier's method the following problem for radial components $p(\gamma_n, r)$, i.e., for eigenforms of the given Sturm-Liouville problem

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}p(\gamma_n,r)+\left(K_1^2-\frac{\gamma_n^2}{r^2}\right)p(\gamma_n,r)=0. \quad (13)$$

The exact solution of this equation shall be obtained in terms of Bessel's functions $J_{\gamma_n}(K_1r)$ and Neumann's functions $N_{\gamma_n}(K_1r)$ in the following form

$$p(\gamma_n,r)=J_{\gamma_n}(K_1r)\dot N_{\gamma_n}(K_1R_1)-N_{\gamma_n}(K_1r)\dot J_{\gamma_n}(K_1R_1) \quad (14)$$

Here the values γ_n shall be considered as the functions of K_1R_1, which, as we have denoted in the first chapter, shall be considered as the solution of a dispersion equation. We shall describe this equation taking into consideration that similarly to the ordinary waveguide dispersion ratio

$$\gamma_n\approx\sqrt{\left(\frac{\omega R_1}{c}\right)^2-\left(\frac{K_1R\pi}{a}\right)^2}$$

$$\dot J_\gamma(K_1R_2)\dot N_\gamma(K_1R_1)-\dot N_\gamma(K_1R_2)\dot J_\gamma(K_1R_1)=0.$$

It has the complex roots at the frequencies lower than the critical ones. The transition of the constant of propagation γ_n to the complex area takes place through the zero as in the direct waveguide, therefore, the critical frequencies of dispersion are given by

$$\dot J_0(K_1R_2)\dot N_0(K_1R_1)-\dot N_0(K_1R_2)\dot J_0(K_1R_1)=0,$$

the solution of which was undertaken above and it is correct for the given case of the superior normal wave.

If the values γ_n form the numerous eigenvalues, the denoted system of functions $p(\gamma_n, r)$ shall form the complete orthogonal system, i.e.,

$$\int_{R_1}^{R_2}p(\gamma_n,r)p(\gamma_m,r)\frac{dr}{r}=0,$$
$$\int_{R_1}^{R_2}p^2(\gamma_n,r)\frac{dr}{r}=N_n\neq0, \quad (15)$$

where N_n is the essence of the norm of the eigenforms.

The selected form of the solution of a problem allows the transition to the traveling waves with the help of Euler's formulae. In this case the angular distribution of pressure may be described in the form of $a_n^{'}e^{i\gamma_n\theta}+b_n^{'}e^{-i\gamma_n\theta}$, this presenting the wave in terms of the incident and reflected waves.

Now let us start with the adjustment of solutions, described differently in the various sections of the waveguide channel. By means of boundary conditions (11), using the solution (12), we shall obtain four equations

$$\frac{1}{r}\sum_{n=1}^{\infty}a_n p(\gamma_n,r)=\rho\omega^2 v_1(r-R_1),$$
$$\frac{1}{r}\sum_{n=1}^{\infty}b_n p(\gamma_n,r)=\rho\omega^2 v_2(r-R_1), \qquad (16)$$
$$\sum_{n=1}^{\infty}\left\{\frac{a_n\cos\gamma_n\varphi_0-b_n}{\gamma_n\sin\gamma_n\varphi_0}\right\}p(\gamma_n,r)=p_1(r-R_1),$$
$$\sum_{n=1}^{\infty}\left\{\frac{a_n-b_n\cos\gamma_n\varphi_0}{\gamma_n\sin\gamma_n\varphi_0}\right\}p(\gamma_n,r)=p_2(r-R_1).$$

At first it is convenient to consider the last two equations. By means of integral transformation with an account of orthogonality of eigenforms $p(\gamma_n,r)$ we shall obtain

$$a_n\cos\gamma_n\varphi_0-b_n=\frac{\gamma_n\sin\gamma_n\varphi_0}{N_n}\int_{R_1}^{R_2}p_1(r-R_1)p(\gamma_n,r)\frac{dr}{r}, \qquad (17)$$

$$a_n-\cos\gamma_n\varphi_0 b_n=\frac{\gamma_n\sin\gamma_n\varphi_0}{N_n}\int_{R_1}^{R_2}p_2(r-R_1)p(\gamma_n,r)\frac{dr}{r}.$$

Solving this system of algebraic equations we shall obtain a_n and b_n. Inserting a_n and b_n in the first pair of equations (12), we shall obtain

$$\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin\gamma_n\varphi_0}\{p_2(r_1-R_1)p(\gamma_n,r)p(\gamma_n,r_1)-$$
$$-\cos(\gamma_n,\varphi_0)p_1(r_1-R_1)p(\gamma_n,r_1)p(\gamma_n,r)\}=\rho\omega^2 v_1(r-R_1) \qquad (18)$$

$$\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin\gamma_n\varphi_0}\{\cos(\gamma_n,\varphi_0)p_2(r_1-R_1)p(\gamma_n,r)p(\gamma_n,r_1)-$$
$$-p_1(r_1-R_1)p(\gamma_n,r_1)p(\gamma_n,r)\}=\rho\omega^2 v_2(r-R_1).$$

The values $p_1(r-R_1)$ and $p_2(r-R_1)$ are self-expressed through $v_1(y)$ and $v_2(y)$ exactly like in the plane wave case [1]. The given system can be presented in the following form

$$\int_0^a v_1(y_1)K_{11}(y_1,r)dy_1+\int_0^a v_2(y_1)K_{12}(y_1,r)dy_1- \qquad (19)$$
$$-\rho\omega^2 v_1(r-R_1)=\Phi_1(r),$$

$$\int_0^a v_1(y_1)K_{21}(y_1,r)dy_1+\int_0^a v_2(y_1)K_{22}(y_1,r)dy_1-$$
$$-\rho\omega^2 v_2(r-R_1)=\Phi_2,$$

where the following notations are introduced:

$$K_{11}(y,r)=\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin(\gamma_n\varphi_0)}p(\gamma_n,r_1)p(\gamma_n,r)M_*,$$

$$K_{12}(y,r)=\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin(\gamma_n\varphi_0)}\cos(\gamma_n\varphi_0)p(\gamma_n,r_1)p(\gamma_n,r)M_*,$$

$$K_{21}(y,r)=\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin(\gamma_n\varphi_0)}p(\gamma_n,r_1)p(\gamma_n,r)M_*,$$

$$K_{22}(y,r)=\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}dr_1}{N_n\sin(\gamma_n\varphi_0)}\cos(\gamma_n,r)p(\gamma_n,r_1)p(\gamma_n,r)M_*,$$

where M_* is expressed in the following form:

$$M_*=\left[\frac{\rho\omega^2}{iK_1a}+\sum_{m=1}^{\infty}\frac{2\rho\omega^2\cos\frac{m\pi(r_1-R_1)}{a}\cos\frac{m\pi y}{a}}{\sqrt{(K_1a)^2-(m\pi)^2}}\right].$$

$$\Phi_1=2\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}\cos(\gamma_n\varphi_0)}{N_n\sin(\gamma_n\varphi_0)}p(\gamma_n,r)p(\gamma_n,r_1)dr_1,$$

$$\Phi_2=2\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1}r_1^{-1}}{N_n\sin(\gamma_n\varphi_0)}p(\gamma_n,r)p(\gamma_n,r_1)dr_1,$$

This system of integral equations can be solved by using the methods of mathematical physics, elaborated for one equation, for example, by means of a resolving or reduction to the system of algebraic equations. The latter method, as it is known, has several variants of realization. The method of change of infinite series in expressions for kernel of integral transformations by finite series by means of truncation is more convenient. This method is known also as the change of exact value of a kernel by its approximation, which leads the kernel to the degenerated form.

Conclusions

When studying normal wave propagation we obtain a convenient system of integral equations for computation of diffraction of these waves. Analysis normal wave diffraction allows the peculiarities of sound wave propagation to be identified in the curved duct section when solving issues of acoustic noise reduction.

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Normaliosios bangos difrakcija iðlenktoje bangolaidþio posûkio dalyje

Reziumë

Metodais, panaðiais á tuos, kurie apraðyti straipsnyje [1], tiriama normalioji banga, sklindanti iðlenktoje pastovaus skerspjûvio bangolaidþio posûkio dalyje. Sprendþiant uþdavinius staèiakampëse arba cilindrinëse koordinatëse, kintamuosius dydþius galima iðskirstyti, todël pradiniam laukui suþadinti m-tàjà bangà galima tirti atskirai.

Kaip banginis laukas sklinda iðlenktoje posûkio dalyje, sprendþiama cilindrinëse koordinatëse, kai kraðtinës sàlygos formuojamos tarp tiesiosios ir iðlenktosios bangolaidþio dalies.

Banginës lygties uþdavinys sprendþiamas kintamø atskyrimo metodu ir iðreiðkiami eilute.

Pagal Eilerio formules parinkta sprendimo forma, leidþia pereiti prie bëganèiø bangø. Laukas vaizduojamas krintanèia ir atspindþio bangomis. Pasinaudojant kraðtinëmis sàlygomis sprendþiamos banginës lygtys jas integruojant. Gaunama patogi integralinið lygèiø sistema, normaliø bangø difrakcijai skaièiuoti.

Normaliøjø bangø difrakcijos skaièiavimai leidþia nustatyti garso bangø sklidimo principus iðlenktoje vamzdþio dalyje sprendþiant triukðmo maþinimo klausimus.

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