Sound power and sound wave radiation by a piston in a curved duct

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Introduction

The solution of the problem consists of two parts. The first part defines the sound power radiated by a piston in an elastic duct, the second part identifies sound wave radiation by a piston in a curved duct.

The problem concerning sound propagation in curved ducts has drawn special attention of numerous researchers. On the one hand, this may be due to the fact that this problem is in the focus since it is the generalisation of the theory of waveguides. On the other hand, solution of tasks related to the propagation of waves in curved ducts is of a practical importance, because almost any system of ducts includes conjugation of straight duct sections by means of curves.

Main difficulties occurring when solving tasks may be traced on the example of the simplest task of sound wave propagation through a duct having a rectangular cross-section with rigid walls in the case its longitudinal axis line curves in circumference. In this case a cylindrical system of coordinates may be used.

Sound power radiated in an elastic duct by piston with arbitrary axial-symmetrical velocity distribution

Let us consider a sound radiation in a duct when in a cross-section z=0 arbitrary distribution of axial velocity $V_z(r)$ is set. We shall denote the sound velocity that is created in a duct, through $p\ (r, z)$. A temporary multiplier exp($-i\omega t$) shall be omitted for a purpose of simplicity.

The radiated sound power is given by

$$P = \pi Re\left\{\int_0^a \left(pV_z^*\right)_{z=0} rdr\right\}. \quad (1)$$

Here the sign (∗) denotes a complex-conjugated quantity.

Here we shall solve a task in the same approximation as in the work [1]:

$$|Z_a| >> |Z_{rad}|,$$

where Z_a is the shell impedance, Z_{rad} - is the radiation impedance. Then the zero root of equation will be just imaginary and small value, i.e., $|\mu_0 a| << 1$. The other roots will be real.

The sound pressure in a duct is:

$$p(r,z) = \rho\omega\sum_{n=0}^{\infty}\frac{V_n}{\sqrt{k_n^2-\mu_n^2}}J_0(\mu_n r)e^{i\sqrt{k^2-\mu_n^2}} . \quad (2)$$

Here $k=\omega/c$ is the wave number; ρ and c are the density and the velocity of a sound wave propagation in a duct; J_0 is the Bessel function of zero order; V_n is the n-th wave amplitude in the velocity distribution $V_z(r)$.

Let us expand the axial component of the velocity $V_z(r)$ in a series at z=0:

$$V_z(r) = \sum V_n J_0(\mu_n r). \quad (3)$$

The amplitudes of the velocity V_n will be defined by ratio:

$$V_n = \frac{2\int_0^a V_z(r)J_0(\mu_n r)rdr}{a^2 J_0^2(\mu_n a)\left(1+\frac{F^2}{\mu_n^2}\right)}. \quad (4)$$

Here $F= -i\rho\omega/Z_a$.

Substituting expressions (3) and (2) into (1), we shall obtain:

$$P = \pi\rho\omega Re\left\{\sum_n\sum_m \frac{V_n V_m^*}{\sqrt{k^2-\mu_n^2}}\int_0^a J_0(\mu_n r)J_0^*(\mu_m r)rdr\right\} \quad (5)$$

As indicated in [2], the functions $J_0(\mu_n r)$, where μ_n is the root of the dispersion equation, form the orthogonal system. In the approximation $|Z_a| >> |Z_{rad}|$ the functions $J_0(\mu_n r)$ and $J_0^*(\mu_n r)$ are also orthogonal. As a result, the integral I in Eq. (5) is equal to 0 at $n\neq m$, and at $n=m$

$$I = \frac{a^2}{2}\left(1+\frac{F^2}{\mu_n^2}\right)J_0^2(\mu_n a). \quad (6)$$

Substituting into Eq. (5) J from (6) and V_n from (4), we shall obtain:

$$P = \frac{2\pi\rho\omega}{a^2} Re\left\{\sum_{n=0}^{\infty}\frac{\left[\int_0^a V_z(r)J_0(\mu_n r)rdr\right]^2}{\sqrt{k^2-\mu_n^2}J_0^2(\mu_n a)\left(1+\frac{F^2}{\mu_n^2}\right)}\right\}. \quad (7)$$

Eq. (7) defines the sound power, which is radiated to the duct, if at the cross-section z=0 the distribution of axial velocities $V_z(r)$ is given.

Let us consider as an example the case when $V_z(r)=V_0 J_0(\alpha r)$. Such a distribution in the first approximation will be observed at vibrating of the membrane, located in the plane z=0.

Then an integral in the Eq. (7)

$$I = V_0 \int_0^a J_0(\alpha r) J_0(\mu_n r) r dr = \\ = V_0 a \frac{\alpha J_1(\alpha a) J_0(\mu_n a) - \mu_n J_0(\alpha a) J_1(\mu_n a)}{\alpha^2 - \mu_n^2}. \quad (8)$$

Substituting expression (8) into (9), we shall obtain:

$$P = \frac{\pi a^2 \rho c V_0^2}{2} \times \\ \times \sum_{n=0}^{\infty} \frac{4\left[\alpha J_1(\alpha a) J_0(\mu_n a) - \mu_n J_0(\alpha a) J_1(\mu_n a)\right]^2}{a^2 \left(\alpha^2 - \mu_n^2\right)^2 \sqrt{1 - \frac{\mu_n^2}{k^2}} \left(1 + \frac{F^2}{\mu_n^2}\right) J_0^2(\mu_n a)}. \quad (9)$$

In Eq. (9) the multiplier before the sign of the total is the sound power which is radiated into a duct by a rigid cylinder, vibrating with the amplitude V_0. Under the sign of the total there is an expression of the n-th wave radiation coefficient.

Sound wave radiation by piston in curved duct

Many studies [3, 4] are devoted to the problem of a sound wave propagation in a system consisting of two straight ducts connected by a bent duct. The simplest solution is obtained when ducts have rectangular cross-section and middle axis line of bent part is changing in circumference. Nevertheless, many difficulties must be overcome in this problem, as for example, solution of infinite system of linear equations what greatly complicates solution and hampers understanding of physical phenomena.

The given study presents the analysis of simpler problem: on one end of bent part (knee) of rectangular cross-section there is a piston which vibrates with the velocity $v = v_0 e^{-i\omega t}$, where $\omega = 2\pi f$ is the angular frequency. The other end of the knee ($\varphi = \pi/2$) is connected with infinite straight duct (Fig. 1). Let us define the sound field p_2, which is created by a piston in a straight duct. Such a problem may be observed in ventilation systems.

Let us assume that in the pipe-line there is a medium characterised by the density ρ_c and the sound propagation velocity c_c. Movement of the medium in the bent part is going to be analysed in the cylindrical system of the coordinates (r, φ, z), where axis z is located perpendicularly to the plane of the drawing of Fig. 1.

As the study [3] reveals, the sound pressure $p_1(r,\varphi,z)$ in a knee having rigid walls may be written as

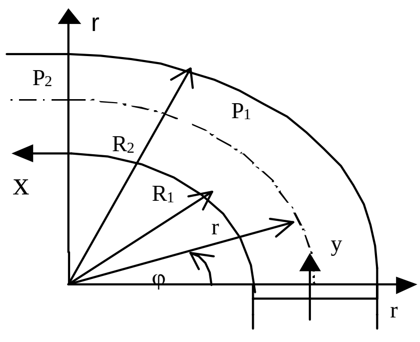

Fig. 1. **Schema of a curved duct**.

$$\left.\begin{aligned} & p_1(r,\varphi,z) v_\varphi = \\ & = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} R_n(k_{rn} r) \cos k_m z \left(\sin v_n \varphi + B_n \cos v_n \varphi\right), \\ & R_n(k_{rn} r) = J v_n(k_{rm} r) - \frac{\dot{J} v_n(k_{rm} R_1)}{\dot{N} v_n(k_{rm} R_1)} N v_n(k_{rn} r). \end{aligned}\right\} \quad (10)$$

Here Jv_n and Nv_n are Bessel and Neumann functions of the order v_n, $k_m = m\pi/d$, where d is the knee width, A_{mn} and B_n are some constant amplitudes, $k_{rm} = \sqrt{k_c^2 - k_m^2}$ is the radial wave number, $k_c = \omega/c_c$ is the medium wave number. The point above Jv_n and Nv_n indicates an derivative. The term exp($-i\omega t$) for the sake simplicity is omitted.

The function R_n in (10) satisfies the boundary conditions of inner ($r=R_1$) and outer ($r=R_2$) walls of the knee. As the walls are considered to be rigid, so the radial oscillation velocity of medium particles at their surface equals to 0. It follows that

$$\dot{R}_n = \dot{J} v_n - \frac{\dot{J} v_n(k_{rm} R_1)}{\dot{N} v_n(k_{rm} R_1)} \dot{N} v_n = 0,$$

when $r=R_1$ and $r=R_2$, or

$$\frac{\dot{J} v(k_{rm} R_1)}{\dot{N} v_n(k_{rm} R_1)} = \frac{\dot{J} v_n(k_{rm} R_2)}{\dot{N} v_n(k_{rm} R_2)}. \quad (11)$$

It is a dispersion equation of the problem under discussion. Here, in contrast to widely known expressions not an argument, but an unknown order of the Bessel function v_n is defined.

A number of algorithms of definition v_n in the Eq. (11) are described. One of them [5] uses the reciprocal procedure and is set by a semi-whole order v_n. In this case functions Jv_n and Nv_n are expressed by elementary functions. By substituting v_n to (11), frequencies ($k_{rm}R_1$), which satisfy the equation are defined. Diagrams and tables which may be used for calculation are given in study [5]

In accordance with the Euler equation the tangent oscillation velocity of medium particles equals

$$v_\varphi = \frac{1}{i\rho_c \omega r}\frac{\partial p_1}{\partial \varphi} = \frac{1}{i\rho_c \omega r}\sum_m \sum_n A_{mn} v_n R_n \times \\ \times \cos k_m z \left(\cos v_n \varphi - B_n \sin v_n \varphi\right). \quad (12)$$

The sound pressure $p_2(x,r,z)$ in a straight duct with rigid walls may be written as

$$p_2(x,r,z) = \\ = \sum_{m=0}^{\infty}\sum_{q=0}^{\infty} D_{mq} \cos k_m z \cos k_q (r - R_1) e^{i\sqrt{k_c^2 - k_m^2 - k_q^2 x}}, \quad (13)$$

where $k_q = q\pi/(R_2 - R_1)$ and D_{mq} are the mode amplitudes (m,q). The oscillation velocity of medium particles along the x direction equals

$$v_x(x,r,z) = \frac{1}{i\rho_c \omega}\frac{\partial p_2}{\partial x} = \\ = \frac{1}{\rho_c \omega}\sum_m \sum_q D_{mq}\sqrt{k_c^2 - k_m^2 - k_q^2} \times \\ \times \cos k_m z \cos k_q (r - R_1) e^{i\sqrt{k_c^2 - k_m^2 - k_q^2}}. \quad (14)$$

The sound pressure and oscillation velocity must satisfy the boundary conditions from which unknown amplitudes A_{mn}, B_n, and D_{mq} are defined:

$$\left.\begin{aligned} &\varphi = 0, v_\varphi(r,\varphi,z) = v_0, \\ &\varphi = \frac{\pi}{2},\ p_1(r,\varphi,z) = p_2(0,r,z), v_\varphi(r,\varphi,z) = v_x(0,r,z) \end{aligned}\right\} \quad (15)$$

By substituting v_φ from (12) as well as p_1 and p_2 from (10) and (13) and v_x from (14) to (15) we get:

$$\left.\begin{aligned} &\frac{1}{i\rho_c\omega r}\sum_m\sum_n A_{mn} v_n R_n(k_{rm} r)\cos k_m z = v_0, \\ &\sum_m\sum_n A_{mn} R_n(k_{rm} r)\cos k_m z = \\ &= \sum_m\sum_q D_{mq}\cos k_m z \cos k_q(r - R_1), \\ &\frac{i}{r}\sum_m\sum_n A_{mn} v_n R_n(k_{rm} r)\cos k_m z B_n = \\ &= \sum_m\sum_q D_{mq}\sqrt{k_c^2 - k_m^2 - k_q^2}\cos k_m z \cos k_q(r - R_1). \end{aligned}\right\} \quad (16)$$

Before starting to solve the system (16) we must remind that functions $R_n(k_{rn}r)$ are orthogonal, i.e., satisfy conditions

$$\int_{R_1}^{R_2} R_n(k_{rm} r) R_j(k_{rm} r)\frac{dr}{r} = \begin{cases} 0 & when\ n \neq j, \\ I_{nn} & when\ n = j, \end{cases}$$

where

$$I_{nm} = \int_{R_1}^{R_2} R_n^2(k_{rm} r)\frac{dr}{r}. \quad (17)$$

From the first equation (16) we may define A_{mn}. For this we multiply both parts of the equation by $R_j \cos kzdrdz$ and integrate within the limits from R_1 to R_2 by r and from 0 to d by z:

$$\begin{aligned} &\frac{1}{i\rho_c\omega}\sum_m\sum_n A_{mn} v_n \int_{R_1}^{R_2} R_n R_j \frac{dr}{r}\int_0^d \cos k_m z \cos k_q z dz = \\ &= v_0 \int_{R_1}^{R_2} R_j dr \int_0^d \cos k_q z dz. \end{aligned} \quad (18)$$

The last integral on the right in (18) changes to 0 at all $q \neq 0$. At $q=0$ it equals to d. Therefore, all $A_{mn}=0$ if $m \neq 0$.

On the basis of orthogonality R_n finally we get

$$A_{0n} = \frac{i v_0 \rho_c \omega I_n}{v_n I_{nn}}, \quad (19)$$

where

$$I_n = \int_{R_1}^{R_2} R_n(k_c r) dr. \quad (20)$$

As $m=0$, so the second equation in (18) may be written

$$\sum_n A_{0n} R_n = \sum_q D_{0q}\cos k_q(r - R_1).$$

Let us multiply both its parts by $\cos k_j(r-R_1)$ and integrate by r from R_1 to R_2. Then, taking in account of orthogonality of the function $\cos k_q(r-R_1)$, we get:

$$D_{0n} = \frac{2 i v_0 \rho_c \omega}{d}\sum_{n=0}^{\infty}\frac{1}{v_n}\frac{I_n I_{nq}}{I_{nn}}, \quad (21)$$

where

$$\left.\begin{aligned} &I_{nq} = \int_{R_1}^{R_2} R_n(k_c r)\cos k_q(r - R_1) dr, \\ &I_{nn} = R_n^2(k_c r)\frac{dr}{r}. \end{aligned}\right\} \quad (22)$$

Full sound pressure in a straight duct equals

$$p_2(x,r) = \\ = \frac{2 i v_0 \rho_c \omega}{d}\sum_{q=0}^{\infty}\cos k_q(r - R_1) e^{i\sqrt{k_c^2 - k_q^2}x} \times \sum_{n=0}^{\infty}\frac{I_n I_{nq}}{v_n I_{nn}}. \quad (23)$$

From (23) it follows that at low frequencies up to the frequency of the first resonance $f_1 = c_c/(2d)$ only a plane wave may propagate in a duct:

$$p_{20}(x,r) = \frac{2 i v_0 \rho_c \omega}{d} e^{i k_c x}\sum_{n=0}^{\infty}\frac{I_n^2}{v_n I_{nn}}. \quad (24)$$

Amplitudes of other waves decay by the law $e^{-\sqrt{k_q^2 - k_c^2}x}$.

As it is known, the piston vibrating with the amplitude v_0, radiates a sound wave in a straight duct having a pressure

$$p_0(x) = \rho_c c_c v_0. \quad (25)$$

The difference of sound pressure levels equals:

$$\Delta L = L_{20} - L_0 = 20\lg\left|\frac{2k_c}{d}\sum_{n=0}^{\infty}\frac{I_n^2}{v_n I_{nn}}\right|,$$

where $L_{20} = 20\lg\left|\frac{p_{20}}{p_c}\right|$, $L_{20} = 20\lg\left|\frac{p_0}{p_c}\right|$, (26)

and $p_c = 2\cdot 10^{-5}$ *Pa* is the reference level of a sound pressure. Eq. (26) shows the difference of a plane wave radiation by a piston to a straight duct through the knee relatively to a sound radiation by a piston directly to a straight duct.

Conclusions

The result obtained shows how the sound wave is radiated by the piston in the elastic and curved duct. This analysis may be applied for reducing a noise radiation in a curved duct.

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Garso galia ir garso bangø iðspinduliavimas stûmokliu á iðlenktà vamzdþio dalá (alkûnæ)

Reziumë

Darbe nustatyta garso galia, iðspinduliuojama stûmokliu á tamprø vamzdá. Antroje dalyje tiriamas garso bangø iðspinduliavimas stûmokliu á iðlenktà vamzdþio dalá.

Pirmuoju atveju tiriama, kada garsas iðspinduliuojamas á vamzdá skerspjûvyje z=0, laisvai pasirenkant aðiná greitá V_z (r). Uþdavinys sprendþiamas cilindrinëse koordinatëse ir gaunama garso galia, kurià á vamzdá iðspinduliuoja standus stûmoklis, svyruojantis amplitude Vo.

Antroje straipsnio dalyje tiriamas garso bangø iðspinduliavimas á alkûnæ. Viename alkûnës gale yra stûmoklis, kuris svyruoja greièiu $V=V_o e^{-i\omega t}$, antras alkûnës galas sujungtas su begaliniu tiesiu vamzdþiu. Nustatomas garso laukas p_2, kuris sukuriamas stûmokliu á vamzdá. Tariame, kad vamzdyje yra terpë, kurios tankis ρ_c, o garso sklidimo joje greitis c_c.

Uþdavinys taip pat sprendþiamas cilindrinëse koordinatëse. Gautas uþdavinio sprendimo rezultatas rodo, kaip keièiasi plokðèiosios bangos iðspinduliavimas stûmokliu á alkûnæ ir á vamzdþio tiesiàjà dalá. Gauti rezultatai gali bûti panaudoti per alkûnæ pereinanèio garso susilpnëjimui nustatyti.