Normal wave diffraction in the rounded section of a waveguide

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Introduction

An issue of sound wave propagation in curved ducts has attracted attention of numerous researchers. The problem is of serious interest since it is the generalization of the theory of waveguides, Solution of problems related to wave propagation in curved ducts are of special practical importance, since almost any piping system includes conjugations of straight sections by means of curved ones. Normal wave propagation in the curved bend section of the waveguide will be studied using methods analogous to those as in our paper [1]. Plane wave propagation in the elbow bend was investigated in [1]. Plane wave sound energy is directed towards longitudinal axis along the front.

Therefore it is reasonable to analyse issues concerning curved ducts with regard to the application of mathematical methods for normal waves.

It should be noted that there are other methods for computation of sound propagation in curvilinear waveguides. In some works, for example [2,3], a general method for studying of heterogeneous waveguides – a method of transversal cross section – is applied. However, in a general case a rather bulky algorithm, requiring the solution of an infinite system of differential equations, enables to obtain numerical results.

Theory of normal wave diffraction in curved ducts

Further we shall study a normal wave propagating along a waveguide with a constant cross-section. Due to the separability of variables in both rectangular and cylindrical coordinates it is possible to see the single wave of m-th order as an initial exciting field:

$$p_i = \cos\frac{m\pi z}{b}\, e^{ix_1\sqrt{K^2-\left(\frac{m\pi}{b}\right)^2}-i\omega t} . \qquad (1)$$

Further the *m* index shall be omitted, taking into account the separability of the corresponding coordinates and accepting the above-introduced designations. Thus the disseminated field in the semi-finite rectilinear section p_2 shall be written without the index *m* in the form of

$$p_2 = \alpha_0 e^{-iK_1x_1-i\omega t} + \sum_1^{\infty}\alpha_n \cos\frac{n\pi y}{a} e^{-ix_1\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}-i\omega t} \qquad (2)$$

and the one passing through the rounded section of the part in the form of p_t

$$p_t = \beta_0 e^{iK_2x_2-i\omega t} + \sum_1^{\infty}\beta_n \cos\frac{n\pi y_2}{a} e^{ix_2\sqrt{K_2^2-\left(\frac{n\pi}{a}\right)^2}-i\omega t} \qquad (3)$$

$$K_2 = K_1 = \sqrt{\frac{\omega^2}{c^2}-\left(\frac{m\pi}{b}\right)^2} .$$

Here everywhere the above-introduced notations and the condition are used

$$\frac{\omega^2}{c^2} > \frac{m^2\pi^2}{b^2},\ J_m\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2} > 0. \qquad (4)$$

which expresses the possibility of realization of the *m*-th mode in the semi-infinite waveguide, i.e., the non-zero condition of excitation.

As in the previous case instead of the coefficients we shall search for $p_1(y)$ and $v_1(y)$. From the boundary conditions at the joint of the curvilinear and the first rectilinear sections

$$\rho_1\omega^2 v_1(y) = \frac{\partial p_1}{\partial x_1} \qquad (5)$$

and on the other side

$$\frac{\partial p_1}{\partial x} = iK_1(1-\alpha_0) + \sum_1^{\infty}\alpha_n\left[-i\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}\right]\cos\frac{n\pi y}{a} \qquad (6)$$

then with the help of Fourier's theorem we shall obtain

$$1-\alpha_0 = \frac{\rho_1\omega^2}{iK_1a}\int_0^a v_1(y)dy,$$

$$\alpha_n = \frac{2\rho\omega^2}{a\left[-i\sqrt{K_1^2-\left(\frac{n\pi}{a}\right)^2}\right]}\int_0^a v_1(y)\cos\frac{n\pi y}{a}dy,$$

and, consequently

$$\frac{\partial p_1}{\partial x_1} = \rho\omega^2 v_1(y,) \qquad (7)$$

$$p_1(y) = 2 - \int_0^a v_1(y)\left\{\frac{\rho\omega^2}{iK_1a} + \sum_1^{\infty}\frac{2\rho\omega^2}{i\sqrt{K_1^2a^2-(n\pi)^2}}\times\right.$$

$$\left.\times\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}\right\}dy.$$

In a similar way, instead of the system of transmission coefficients β_n we shall introduce the functions $p_2(y)$ and $v_2(y)$ /the distribution of pressure and velocity of particles in the second cross-section, i.e., at the joint of the curvilinear section and the second semi-waveguide/. From the Euler's equation [1] we shall find coefficients β_n, expressing them in terms of $v_2(y)$:

$$\beta_0 = \frac{\rho\omega^2}{iK_1 a}\int_0^a v_2(y)dy, \tag{8}$$

$$\beta_n = \frac{2\rho\omega^2}{i\sqrt{(K_1 a)^2 - (n\pi)^2}}\int_0^a v_2(y)\cos\frac{n\pi y}{a}dy,$$

and, finally, we describe

$$\frac{\partial p_2}{\partial x_2} = \rho\omega^2 v_2(y), \tag{9}$$

$$p_2(y) = \int_o^a v_2(y)\left[\frac{\rho\omega^2}{iK_1 a} + \sum_1^\infty \frac{2\rho\omega^2\cos\frac{n\pi y}{a}\cos\frac{n\pi y_1}{a}}{i\sqrt{(K_1 a)^2 - (n\pi)^2}}\right]dy_1.$$

The equations (7)-(9) enable the boundary conditions for a problem on a wave field in the curvilinear section of a waveguide to be described. The field in that section must satisfy a wave equation, described in the cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} + K_1^2 p = 0 \tag{10}$$

at the following boundary conditions

$$\left.\begin{aligned} p(r,0) &= p_1(r - R_1) \\ \frac{1}{r}\frac{\partial p}{\partial\theta} &= \rho\omega^2 v(r - R_1) \end{aligned}\right\}\theta = 0,$$
$$\left.\begin{aligned} p(r,\varphi_0) &= p_2(r - R_1) \\ \frac{1}{r}\frac{\partial p}{\partial\theta} &= \rho\omega^2 v_2(r - R_1) \end{aligned}\right\}\theta = \varphi_0. \tag{11}$$

Here φ_0 is the angle of bending of the curvilinear section, R_1 is the inferior of the rounding radii, $R_2=R_1+a$ is the outer radius of the rounded part of the waveguide.

As is seen, the boundary conditions for the field p(r,θ,z), in the essence, are the condition for the jointing of the wave fields on the boundaries of the rectilinear and rounded sections of the waveguide.

The solution of the wave equation (10) for the curvilinear section may be described by means of the separation of the variables in the form of the following series

$$p(r,\theta) = \sum_{n=0} p(\gamma_n r)\left\{\frac{a_n\cos\gamma_n(\theta - \varphi_0) - b_n\cos\gamma_n\theta}{\gamma_n\sin\gamma_n\varphi_0}\right\} \tag{12}$$

The multiplier $\cos\frac{mz\pi}{b}$ as it was mentioned, shall be omitted everywhere. Here γ_n are the separation constant of the variables that are forming the series of the eigenvalues [4]. From the wave equation (10) we shall obtain by Fourier's method the following problem for radial components $p(\gamma_n, r)$, i.e., for eigenforms of the given Sturm-Liouville problem

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}p(\gamma_n,r) + \left(K_1^2 - \frac{\gamma_n^2}{r^2}\right)p(\gamma_n,r) = 0 \tag{13}$$

The exact solution of this equation shall be described by means of Bessel's functions $J_{\gamma_n}(K_1 r)$ and Neumann's functions $N_{\gamma_n}(K_1 r)$ in the following form

$$p(\gamma_n,r) = J_{\gamma_n}(K_1 r)\dot{N}_{\gamma_n}(K_1 R_1) - N_{\gamma_n}(K_1 r)\dot{J}_{\gamma_n}(K_1 R_1). \tag{14}$$

Here the values γ_n shall be considered as the functions of K_1R_1 which, as we have denoted in the first chapter, shall be considered as the solution of a dispersion equation. We shall describe this equation taking into consideration that similarly to the ordinary waveguide dispersion ratio $\gamma_n \approx \sqrt{\left(\frac{\omega R_1}{c}\right)^2 - \left(\frac{K_1 R\pi}{a}\right)^2}$,

$$\dot{J}_\gamma(K_1 R_2)\dot{N}_\gamma(K_1 R_1) - \dot{N}_\gamma(K_1 R_2)\dot{J}_\gamma(K_1 R_1) = 0.$$

It has the complex roots in the frequencies lower than the critical ones. The transition of the constant of propagation γ_n to the complex area takes place through the zero as in the direct waveguide, therefore the critical frequencies are given by

$$\dot{J}_0(K_1 R_2)\dot{N}_0(K_1 R_1) - \dot{N}_0(K_1 R_2)\dot{J}_0(K_1 R_1) = 0,$$

the solution of which was undertaken above and it is good for the given case of the higher or normal wave.

If the values γ_n form the numerous eigenvalues, the detected system of functions $p(\gamma_n,r)$ shall form the complete orthogonal system, i.e.,

$$\int_{R_1}^{R_2} p(\gamma_n,r)p(\gamma_m,r)\frac{dr}{r} = 0,$$
$$\int_{R_1}^{R_2} p^2(\gamma_n,r)\frac{dr}{r} = N_n \neq 0, \tag{15}$$

where N_n is the essence of the norm of the eigenforms.

The selected form of the solution of a problem allows the transition to the traveling waves with the help of the Euler's formulae. In this case the angular distribution of pressure may be described in the form of $a_n^{\copyright} e^{i\gamma_n\theta} + b_n^{\copyright} e^{-i\gamma_n\theta}$, thus presenting the wave in terms of the incident and reflected waves.

Now let us start with the adjustment of solutions, described differently in the various sections of the waveguide channel. By means of the boundary conditions (11) using the solution (12) we shall obtain four equations:

$$\frac{1}{r}\sum_{n=1}^\infty a_n p(\gamma_n,r) = \rho\omega^2 v_1(r - R_1),$$
$$\frac{1}{r}\sum_{n=1}^\infty b_n p(\gamma_n,r) = \rho\omega^2 v_2(r - R_1), \tag{16}$$
$$\sum_{n=1}^\infty\left\{\frac{a_n\cos\gamma_n\varphi_0 - b_n}{\gamma_n\sin\gamma_n\varphi_0}\right\}p(\gamma_n,r) = p_1(r - R_1),$$
$$\sum_{n=1}^\infty\left\{\frac{a_n - b_n\cos\gamma_n\varphi_0}{\gamma_n\sin\gamma_n\varphi_0}\right\}p(\gamma_n,r) = p_2(r - R_1).$$

At first it is convenient to consider the last two equations. By means of integral transformation with an account of orthogonality of the eigenmodes $p(\gamma_n, r)$ we shall obtain

$$a_n \cos\gamma_n\varphi_0 - b_n = \frac{\gamma_n \sin\gamma_n\varphi_0}{N_n}\int_{R_1}^{R_2} p_1(r-R_1)p(\gamma_n, r)\frac{dr}{r},$$

$$a_n - \cos\gamma_n\varphi_0 b_n = \frac{\gamma_n \sin\gamma_n\varphi_0}{N_n}\int_{R_1}^{R_2} p_2(r-R_1)p(\gamma_n, r)\frac{dr}{r}. \tag{17}$$

By solving this system of algebraic equations we shall obtain a_n and b_n. Inserting a_n and b_n in the first pair of equations (12), we shall obtain

$$\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin\gamma_n\varphi_0}\{p_2(r_1-R_1)p(\gamma_n, r)p(\gamma_n, r_1) -$$

$$-\cos(\gamma_n, \varphi_0)p_1(r_1-R_1)p(\gamma_n, r_1)p(\gamma_n, r)\} = \rho\omega^2 v_1(r-R_1), \tag{18}$$

$$\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin\gamma_n\varphi_0}\{\cos(\gamma_n, \varphi_0)p_2(r_1-R_1)p(\gamma_n, r)p(\gamma_n, r_1) -$$

$$-p_1(r_1-R_1)p(\gamma_n, r_1)p(\gamma_n, r)\} = \rho\omega^2 v_2(r-R_1).$$

The values $p_1(r\text{-}R_1)$ and $p_2(r\text{-}R_1)$ are self-expressed through $v_1(y)$ and $v_2(y)$ exactly like in the plane wave case [1]. The given system is described in the following form

$$\int_0^a v_1(y_1)K_{11}(y_1, r)dy_1 + \int_0^a v_2(y_1)K_{12}(y_1, r)dy_1 - \tag{19}$$

$$-\rho\omega^2 v_1(r-R_1) = \Phi_1(r),$$

$$\int_0^a v_1(y_1)K_{21}(y_1, r)dy_1 + \int_0^a v_2(y_1)K_{22}(y_1, r)dy_1 -$$

$$-\rho\omega^2 v_2(r-R_1) = \Phi_2(r).$$

where the following notations are introduced:

$$K_{11}(y, r) = \int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n\varphi_0)}p(\gamma_n, r_1)p(\gamma_n, r)M_*,$$

$$K_{12}(y, r) = \int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n\varphi_0)}\cos(\gamma_n\varphi_0)p(\gamma_n, r_1)p(\gamma_n, r)M_*,$$

$$K_{21}(y, r) = \int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n\varphi_0)}p(\gamma_n, r_1)p(\gamma_n, r)M_*,$$

$$K_{22}(y, r) = \int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1} dr_1}{N_n \sin(\gamma_n\varphi_0)}\cos(\gamma_n, r)p(\gamma_n, r_1)p(\gamma_n, r)M_*,$$

Here M_* is expressed in the following form:

$$M_* = \left[\frac{\rho\omega^2}{iK_1 a} + \sum_{m=1}^{\infty}\frac{2\rho\omega^2\cos\frac{m\pi(r_1-R_1)}{a}\cos\frac{m\pi y}{a}}{\sqrt{(K_1 a)^2-(m\pi)^2}}\right].$$

$$\Phi_1 = 2\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1}\cos(\gamma_n\varphi_0)}{N_n \sin(\gamma_n\varphi_0)}p(\gamma_n, r)p(\gamma_n, r_1)dr_1,$$

$$\Phi_2 = 2\int_{R_1}^{R_2}\sum_{n=1}^{\infty}\frac{\gamma_n r^{-1} r_1^{-1}}{N_n \sin(\gamma_n\varphi_0)}p(\gamma_n, r)p(\gamma_n, r_1)dr_1.$$

This system of integral equations can be solved by using the methods of mathematical physics, elaborated for one equation, for example, by means of resolving or reduction to the system of algebraic equations. The latter method, as it is known, has several variants of implementation. The method of change of infinite series in expressions for kernel of integral transformations by finite series (by means of truncation) is more convenient. This method is known also as the change of exact value of a kernel by its approximation, which leads the kernel to the degenerated form.

Conclusions

For studying of propagation of normal wave we obtained a convenient system of integral equations for computation of diffraction of these waves. Normal wave diffraction computation allows the principles of sound wave propagation to be identified in the curved duct section when solving issues of sound isolation.

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Normaliosios bangos difrakcija išlenktoje bangolaidžio posūkio dalyje

Reziumė

Metodais, panašiais kaip ir anksčiau skelbtame straipsnyje [1], tiriame normaliąją bangą, sklindančią išlenktoje pastovaus skerspjūvio bangolaidžio posūkio dalyje. Sprendžiant uždavinius stačiakampėse arba cilindrinėse koordinatėse, kintamuosius dydžius galima išskirstyti, todėl pradinį lauką sužadinančią *m*-tąją bangą galima tirti atskirai.

Bangų lauko sklidimo išlenktoje posūkio dalyje uždavinį sprendžiame cilindrinėse koordinatėse, kai kraštinės sąlygos formuojamos tarp tiesiosios ir išlenktosios bangolaidžio dalių.

Banginės lygties uždavinys sprendžiamas kintamųjų atskyrimo metodu ir išreiškiamas eilute.

Pagal Eulerio formules parinkta sprendimo forma leidžia pereiti prie bėgančiųjų bangų. Laukas vaizduojamas krintančiąja ir atsispindinčiąja bangomis. Pasinaudojant kraštinėmis sąlygomis, sprendžiamos banginės lygtys jas integruojant. Gaunama patogi integralinių lygčių sistema normaliųjų bangų difrakcijai apskaičiuoti.

Sprendžiant triukšmo mažinimo klausimus, normaliųjų bangų difrakcijos skaičiavimai leidžia nustatyti garso bangų sklidimo išlenktoje vamzdžio dalyje principus.

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